Lösningsskiss till tentamen i Matematisk analys, del 1, 2020-08-17

1 $f(x) = (x^3-2)e^{3x}$ • $D_f = \mathbb{R}$, $f(x)=0 \Leftrightarrow x = \sqrt[3]{2}$

• $\lim\limits_{x\to-\infty} (x^3-2)e^{3x} = 0$ (pga hastighetstabel) $\Rightarrow y=0$ vågrät asymptot

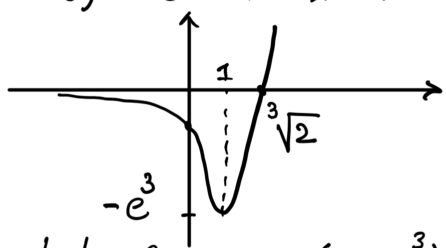
$\lim\limits_{x\to\infty} (x^3-2)e^{3x} = \infty$

• $f'(x) = 3x^2e^{3x} + 3(x^3-2)e^{3x} = 3e^{3x}(x^3+x^2-2) =$

$= 3e^{3x}(x-1)(x^2+2x+2) = 0 \Leftrightarrow x=1$

	$(-\infty,1)$	1	$(1,\infty)$
f'	$-$	0	$+$
f	$\searrow$	min	$\nearrow$

$f(1) = -e^3$



Svar: $y=0$ – vågrät asymptot, lok. min i $(1,-e^3)$

2a $\ln(x+2) + \ln(x-3) = \ln(x+9) \overset{\text{log. lag}}{\Leftrightarrow} \begin{cases} \ln(x+2)(x-3) = \ln(x+9) \\ \text{krav: } x>3 \end{cases} \Leftrightarrow$

$\begin{cases} (x+2)(x-3) = x+9 \\ x>3 \end{cases} \Leftrightarrow x^2-x-6 = x+9,\ x>3 \Leftrightarrow x^2-2x-15=0,\ x>3 \Leftrightarrow$

$/ (x-1)^2-16=0$; $(x-5)(x+3)=0 / \Leftrightarrow \begin{cases} x=5, \\ x=-3 \text{ – falsk rot} \\ x>3 \end{cases}$

Svar: $x=5$

2b $x^2-2x+2=0 \Leftrightarrow x = 1\pm i$

$p(x) = x^4+ax+4$ är delbart med $x^2-2x+2 \Rightarrow p(1+i)=0 \Rightarrow$

$(1+i)^4 + a(1+i)+4 = 0 \Rightarrow \underbrace{(\sqrt{2}\, e^{i\frac{\pi}{4}})^4}_{=-4} + a\sqrt{2}\, e^{i\frac{\pi}{4}} + 4 = 0 \Rightarrow a=0$

För $a=0$ är $p(x) = x^4+4$ och $p(x) = x^4+4 = (x^2+2x+2)(x^2-2x+2) = 0$

$\Leftrightarrow x^2+2x+2=0$ eller $x^2-2x+2=0$

Polynoms nollställen är

$x_1 = +1+i,\ x_2 = +1-i,\ x_3 = -1+i,\ x_4 = -1-i$

Allt. $p(z) = z^4+4 = 0 \Leftrightarrow z^4 = -4$

Sätt $z = re^{i\varphi}$; $-4 = 4e^{i\pi}$. Alltså $z^4 = -4 \Leftrightarrow r^4e^{i4\varphi} = 4e^{i\pi}$

$\Leftrightarrow \begin{cases} r^4 = 4 \\ 4\varphi = \pi + 2\pi k \end{cases},\ k=0,1,2,3 \Leftrightarrow \begin{cases} r = \sqrt{2} \\ \varphi = \frac{\pi+2\pi k}{4}, \end{cases} k=0,1,2,3$

$$z_1 = \sqrt{2}\, e^{i\frac{\pi}{4}} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = 1+i$$

$$z_2 = \sqrt{2}\, e^{i\frac{3}{4}\pi} = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -1+i$$

$$z_3 = \sqrt{2}\, e^{i\frac{5}{4}\pi} = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = -1-i$$

$$z_4 = \sqrt{2}\, e^{i\frac{7}{4}\pi} = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = 1-i$$

Svar: $a=0$; $x_1=1+i$, $x_2=1-i$, $x_3=-1+i$, $x_4=-1-i$

3a

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-5x+4} = \left(\frac{0}{0}\right) = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x-4)} = \lim_{x\to 1} \frac{x+2}{x-4} = \frac{3}{-3} = -1.$$

3b

$$\lim_{x\to\infty} \left(\ln\sqrt{4x^2+x} - \ln x\right) = \lim_{x\to\infty} \ln \frac{\sqrt{4x^2+x}}{x}$$

$$\Big/\ \lim_{x\to\infty} \frac{\sqrt{4x^2+x}}{x} = \lim_{x\to\infty} \sqrt{\frac{4x^2+x}{x^2}} = \lim_{x\to\infty} \sqrt{4+\frac{1}{x}} = 2 \Big/ = \ln 2.$$

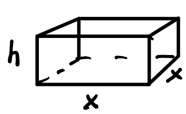
ln är kontinuerlig

3c

$$\lim_{x\to 0} \frac{\ln(1+2x)}{\sin 10x} = \lim_{x\to 0} \underbrace{\cancel{\frac{\ln(1+2x)}{2x}}}_{\to 1} \cdot \frac{2x}{10x} \cdot \underbrace{\cancel{\frac{10x}{\sin 10x}}}_{\to 1} = \frac{2}{10} = \frac{1}{5}.$$

Svar: a) -1 b) $\ln 2$ c) $\frac{1}{5}$.

4



$$A = x^2 + 4x\cdot h = 27 \Rightarrow h = \frac{27-x^2}{4x}$$

$$V = x^2 h = x^2 \cdot \frac{27-x^2}{4x} = \frac{1}{4}(27x - x^3) \to \max, \quad x>0$$

$$V'(x) = \frac{1}{4}(27-3x^2) = \frac{3}{4}(9-x^2) = 0 \Leftrightarrow \begin{array}{l} x=3 \\ x=-3 \notin D_V \end{array}$$

	$(0,3)$	3	$(3,\infty)$
V'	$+$	0	$-$
V	$\nearrow$	max.	$\searrow$

$$V_{max} = V(3) = \frac{3}{4}(27-9) = \frac{3}{4}\cdot 18 = \frac{3}{2}\cdot 9$$

Svar: $V_{max} = V(3) = \frac{27}{2}\ dm^3$.

5 Sätt $f(x) = 2x + 3\ln x - 10\arctan x$, $x>0$

- $D_f = (0,\infty)$, $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = \infty$
- $f'(x) = 2 + \frac{3}{x} - \frac{10}{1+x^2} = \frac{2x(1+x^2) + 3(1+x^2) - 10x}{x(1+x^2)} =$

$$= \frac{2x+2x^3+3+3x^2-10x}{x(1+x^2)} = \frac{2x^3+3x^2-8x+3}{x(1+x^2)} = \frac{(x-1)(2x^2+5x-3)}{x(1+x^2)}$$

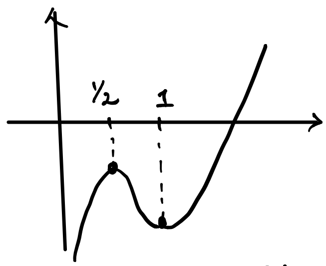
$$= \frac{2(x-1)(x-\frac{1}{2})(x+3)}{x(1+x^2)} = 0 \Leftrightarrow \quad x=1,\ x=\tfrac{1}{2},\ x=-3 \notin D_f$$

	$(0,\frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2},1)$	1	$(1,\infty)$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	e. max	$\searrow$	lok. min	$\nearrow$

$f(\frac{1}{2}) = \underbrace{1-3\ln 2}_{<0} - 10\arctan\frac{1}{2} < 0$

ty $1-3\ln 2 = 1-\ln 8 = \ln e - \ln 8 = \ln\frac{e}{8} < 0$.

$f(1) = 2 - 10\arctan 1 = 2 - 10\cdot\frac{\pi}{4} = 2 - \frac{5}{2}\pi < 0$.



f kont $\Rightarrow$ $f(x)=0$ har en rot $\Rightarrow$

$2x + 3\ln x = 10\arctan x$ har exakt en rot

Svar: 1 rot.

6a $\lim\limits_{x\to 0}\frac{\sin x^2}{x} = \lim\limits_{x\to 0}\underbrace{\frac{\sin x^2}{x^2}}_{\to 1}\cdot\underbrace{x}_{\to 0} = 0 \Rightarrow$ Om $f(0)=0$, så blir f kontinuerlig

6b Antag att $f(x) = \begin{cases} x^2\arctan\frac{1}{x}+2x, & x\neq 0 \\ 0, & x=0 \end{cases}$

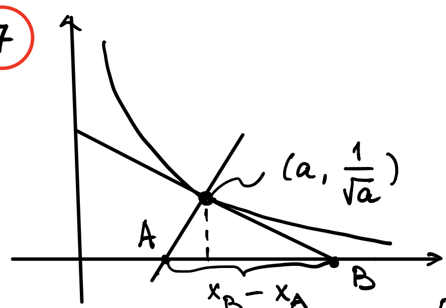
Enligt definition $f'(0) = \lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h} =$

$= \lim\limits_{h\to 0}\frac{h^2\arctan\frac{1}{h}+2h-0}{h} = \lim\limits_{h\to 0}\left(h\arctan\frac{1}{h}+2\right) = 0\cdot\frac{\pi}{2}+2 = 2 \Rightarrow$

$f'(0)$ existerar och $f'(0) = 2$.

Svar: a) om $f(0)=0$, så blir f kont. b) $f'(0)$ existerar och $f'(0)=2$.

7



$y = \frac{1}{\sqrt{x}} \Rightarrow y' = -\frac{1}{2}x^{-3/2}$.

tangenten: $y = \frac{1}{\sqrt{a}} - \frac{1}{2a\sqrt{a}}(x-a)$

normalen: $y = \frac{1}{\sqrt{a}} + 2a\sqrt{a}(x-a)$

B: $\frac{1}{\sqrt{a}} - \frac{1}{2a\sqrt{a}}(x_B-a) = 0 \Rightarrow x_B = 3a$

A: $\frac{1}{\sqrt{a}} + 2a\sqrt{a}(x_A-a) = 0 \Rightarrow x_A = \frac{2a^3-1}{2a^2}$

area $= \frac{1}{2} \cdot \frac{1}{\sqrt{a}}(x_B - x_A) = \frac{1}{2}\frac{1}{\sqrt{a}}\left(3a - \frac{2a^3-1}{2a^2}\right).$

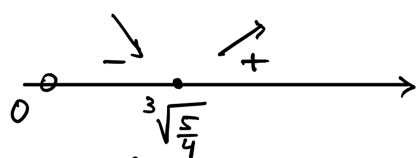
Låt betrakta en funktion

$$A_{\text{area}}(x) = \frac{1}{2}\frac{1}{\sqrt{x}}\left(3x - \frac{2x^3-1}{2x^2}\right), \quad x > 0$$

$$\Rightarrow \quad A(x) = \frac{4x^3+1}{4x^{5/2}} \longrightarrow \underset{x>0}{\text{extr}}$$

$$A'(x) = \frac{12x^2 \cdot 4x^{5/2} - 10x^{3/2}(4x^3+1)}{16x^5} = \frac{2x^{3/2}(24x^3 - 20x^3 - 5)}{16x^5} =$$

$$= \frac{4x^3-5}{8x^{7/2}} = 0 \iff x = \sqrt[3]{\frac{5}{4}}.$$



lok. min

$$A\left(\sqrt[3]{\tfrac{5}{4}}\right) = \frac{4 \cdot \frac{5}{4} + 1}{4 \cdot \left(\frac{5}{4}\right)^{5/6}} = \frac{3}{2}\left(\frac{4}{5}\right)^{5/6}$$

Dessutom,

$$\lim_{x \to 0^+} A(x) = \infty, \quad \lim_{x \to \infty} A(x) = \infty;$$

Svar: A_{max} existerar ej

$$A_{min} = A\left(\sqrt[3]{\tfrac{5}{4}}\right) = \frac{3}{2}\left(\frac{4}{5}\right)^{5/6} \quad \text{v.e.}$$